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EXACT SOLUTIONS OF THE EQUATIONS OF MOTION FOR AN INCOMPRESSIBLE VISCOELASTIC MAXWELL MEDIUM

V. V. Pukhnachev

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The unsteady plane–parallel motion of a incompressible viscoelastic Maxwell medium with constant relaxation time is considered. The equations of motion of the medium and the rheological relation admit an extended Galilean group. The class of solutions of this system which are partially invariant with respect to the subgroup of the indicated group generated by translation and Galilean translation along one of the coordinate axes is studied. The system does not have invariant solutions, and the set of partially invariant solutions is very narrow. A method for extending the set of exact solutions is proposed which allows finding solutions with a nontrivial dependence of the stress tensor elements on spatial coordinates. Among the solutions obtained by this method, the solutions describing the deformation of a viscoelastic strip with free boundaries is of special interest from a point of view of physics.

Key words: *viscoelastic medium, incompressibility, Maxwell relation, Galilean group, partially invariant solution, free boundary motion.*

Introduction. The impetus for writing the present paper came from studies [1–3], which have shown that usual fluids similar to water exhibit viscoelastic properties under conditions where the determining factors are viscosity and shear elasticity while the compressibility of the medium and temperature nonuniformity can be ignored. As a model for such media Korenchenko and Beskachko [3] proposed a mathematical model of an incompressible Maxwell medium with constant relaxation time. The author of the current paper is not aware of systematic studies of the properties of this model have not been performed, while the theory of a compressible viscoelastic Maxwell medium is well developed [4, 5].

As is known, the limiting transition from a slightly compressible medium to an incompressible medium is not trivial even in the linear theory of elasticity. In the case of a Maxwell body, the situation is complicated by the fact that the equations of motion contain a large number of unknown functions. In addition, the limiting system corresponding to an incompressible body loses an important property such as hyperbolicity and have no definite type. In this situation, it is of particular importance to obtain exact solutions that depend on at least two variables. The purpose of the present paper is to solve this problem.

1. Mathematical Model. Below t denotes time, u, v are projections of the velocity vector $\boldsymbol{v}$ onto the Cartesian x and y axes, p is the pressure, $P_{xx} = A$, $P_{xy} = P_{yx} = B$, and $P_{yy} = C$ are the stress tensor elements P, D is the strain rate tensor, I is the unit tensor, τ is the relaxation time, μ is the dynamic viscosity, and ρ is the density of the medium.

The mathematical model considered consists of the Maxwell rheological relation [6]

$$\tau\Big(\frac{\partial P}{\partial t} + \boldsymbol{v}\cdot\nabla P - W\cdot P + P\cdot W\Big) + P = -pI + 2\mu D, \tag{1}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; pukhnachev@gmail.com. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 2, pp. 16–23, March–April, 2009. Original article submitted January 9, 2008.

the momentum equation

$$\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\Big) = \operatorname{div} P, \tag{2}$$

and the continuity equation

$$\nabla\cdot\boldsymbol{v} = 0. \tag{3}$$

Here the symbols ∇ and div denote the gradient and divergence over the variables x and y and the tensor W is the antisymmetric part of the tensor $\nabla\boldsymbol{v}$. The expression in brackets in relation (1) is the rotational Jaumann derivative [6]. The presence of terms with the tensor W in (1) provides invariance of this equation with respect to the rotation group.

Model (1)–(3) applies to finite but relatively small strains and moderate velocities of motion, which allows one to ignore the term responsible for kinetic-energy dissipation in the heat-transfer equation. If there is no heat inflow from the boundaries of the flow region, its motion can be considered isothermal and the quantities τ, μ, and ρ can be considered positive constants. Then, system (1)–(3), consisting of six scalar equations and containing six unknown functions u, v, p, A, B, and C, becomes closed.

As noted above, system (1)–(3) have no definite type and is evolutional with respect to all required functions, except for pressure. In this sense, it is similar to the system of Navier–Stokes equations for an incompressible fluid. However, unlike the latter, which is a nearly parabolic system, system (1)–(3) has a number of properties characteristic of hyperbolic systems. Investigation of general initial-boundary-value problems for the above system is beyond the scope of the present work. At the same time, some characteristic properties of this system can be found by studying its fairly informative exact solutions. A universal tool for the regular construction of the exact solutions is the group analysis of differential equations [7]. Although the group of transformations admitted by system (1)–(3) has not been calculated, it is known *a priori* that it is broad enough.

2. Example of Partially Invariant Solution of System (1)–(3). Direct check of system (1)–(3) shows that it admits an extended Galilean group in the plane (the need to extend the group is due to the presence of the additional unknown functions A, B, and C in this system). The extended Galilean group consists of translations along the t, x, and y axes, Galilean translations along the x and y axes, and conformal rotations in the planes (x, y) and (u, v) in which the tensor P is transformed in a natural manner.

Let us consider the subgroup of the extended Galilean group generated by the translation operators ∂_x and the Galilean translation $t\,\partial_x + \partial_u$ on the x axis. This subgroup has the following complete set of invariants: t, y, v, p, A, B, and C. Since the rank of the corresponding Jacobi matrix (see [7]) is smaller than the number of unknown functions in system (1)–(3), this system does not have an invariant solution with respect to the indicated subgroup. However, its partially invariant solution can be found using the algorithm described in [7]. For this, it is necessary to set

$$v = v(y,t), \quad p = p(y,t), \quad A = A(y,t), \quad B = B(y,t), \quad C = C(y,t) \tag{4}$$

and to find the function $u(x, y, t)$ from the compatibility condition for the overdetermined system obtained from system (1)–(3) by adding the additional equalities

$$v_x = 0, \quad p_x = 0, \quad A_x = 0, \quad B_x = 0, \quad C_x = 0.$$

Substitution of the expression for v from (4) into the continuity equation (3) shows that the function u can only be a linear function of x. For simplicity, we assume that this function is homogeneous. Then, relation (3) implies the equality

$$u = -xv_y. \tag{5}$$

Equality (5) is in fact the result of integration of the automorphic system [7], which, in this case, consists of one equation. Substitution of expression (5) into Eqs. (1) and (2), in view of (4), yields

$$\begin{gathered}
\tau(A_t + vA_y - xv_{yy}B) + A = -p - 2\mu v_y, \\
\tau[B_t + vB_y + xv_{yy}(A - C)/2] + B = -\mu x v_{yy}, \\
\tau(C_t + vC_y + xv_{yy}B) + C = -p + 2\mu v_y, \\
\rho x(-v_{yt} + v_y^2 - vv_{yy}) = B_y, \qquad \rho(v_t + vv_y) = C_y.
\end{gathered} \tag{6}$$

System (6) admits partial separation of variables, which allows one to write the equations of the resolving system and to integrate them in explicit form. Omitting intermediate calculations, the general solution of this system is given by

$$v = -\frac{\alpha y}{1+\alpha t} + l(t), \quad C = \frac{\rho\alpha^2 y^2}{(1+\alpha t)^2} + \rho\Big(\dot l - \frac{\alpha l}{1+\alpha t}\Big) y + m(t), \quad B = \beta\,\mathrm{e}^{-t/\tau},$$
$$p = \rho y^2\Big(\frac{4\tau\alpha^3}{(1+\alpha t)^3} - \frac{\alpha^2}{(1+\alpha t)^2}\Big) + \rho y\Big[-\tau\Big(\ddot l - \frac{\alpha \dot l}{1+\alpha t} + \frac{5\alpha^2 l}{(1+\alpha t)^2}\Big) - \dot l + \frac{\alpha l}{1+\alpha t}\Big] - \tau\dot m - m - \frac{2\alpha\mu}{1+\alpha t}, \tag{7}$$

where α and β are arbitrary constants and l and m are arbitrary functions of t; the dot above the function dependent only on time denotes its derivative. Given the functions v and p, the function $A(y,t)$ is found by solving the linear transport equation

$$\tau(A_t + vA_y) + A = -p - 2\mu v_y.$$

Given arbitrariness in solution (7), it is possible to satisfy the conditions on the free boundaries $\pm y = s(t) \equiv s_0(1+\alpha t)^{-1}$, where $s_0 = \mathrm{const}$. Indeed, if $l = 0$, the kinematic condition $\dot s = v(s,t)$ is satisfied on these lines. Setting $\beta = 0$ leads to the condition of no shear stress $P_{xy} = 0$ on them. If one sets $m(t) = -\rho\alpha^2 s_0^2(1+\alpha t)^{-4}$, the condition of no normal stress $P_{yy} = 0$ is satisfied on the lines $y = \pm s(t)$. This allows a simple physical interpretation of solution (7) with $l = 0$, $\beta = 0$, and the function $m(t)$ specified above. At the initial time, the medium occupies a strip $|y| < s_0$ and has velocity fields $u = \alpha x$ and $v = -\alpha y$ and stress fields $P_{xx} = P_0(y)$, $P_{xy} = 0$, and $P_{yy} = \rho\alpha^2(y^2 - s_0^2)$, where P_0 is an even function of y from the class $C^1[-s_0, s_0]$. The further motion occurs by inertia, and the lines $y = \pm s(t)$ remain free boundaries. For $\alpha > 0$, the solution is defined for all $t > 0$ and the strip width decreases with time as t^{-1}. For $\alpha < 0$, collapse occurs at the time $t^* = -\alpha^{-1}$: the extending strip occupies the entire plane. The cause of this behavior of the medium is the unlimited growth of the longitudinal velocity components as $|x| \to \infty$.

If one sets $\tau = 0$ in the constructed solution, it becomes the solution of the problem of symmetric deformation of a viscous incompressible fluid strip with a linear velocity field [8]. Setting $\mu = 0$ in the last solution, we obtain the well-known Ovsyannikov solution for an ideal fluid ("beam under a punch") [9].

3. Extension of the Family of Exact Solutions. A characteristic feature of the solution of system (1)–(3) constructed in Sec. 2 is that the velocities in it depend linearly on the spatial coordinates, and the stress field does not depend on the variable x. This does not permit one to use this solution as the basis, to consider, for example, the problem of the deformation of a strip by tension or compression forces in the x direction. However, the set of exact solutions can be considerably extended by weakening the requirement of the invariance of some of the required functions for translations and Galilean translations along the x axis. We set

$$v = v(y,t), \quad P_{xx} = D(x,y,t), \quad P_{xy} = E(x,y,t), \quad P_{yy} = C(y,t), \quad p = p(y,t). \tag{8}$$

Thus, the functions v, P_{yy}, and p are still invariants of the indicated two-parameter subgroup of the extended Galilean group whereas P_{xx} and P_{xy} are not such invariants.

Confining ourselves, as earlier, to a homogeneous dependence of the function u on the variable x and substituting relations (5) and (9) into Eqs. (1) and (2), we have

$$\tau(D_t - xv_yD_x + vD_y - xv_{yy}E) + D = -p - 2\mu v_y,$$
$$\tau[E_t - xv_yE_x + vE_y + xv_{yy}(D-C)/2] + E = -\mu x v_{yy},$$
$$\tau(C_t + vC_y + xv_{yy}E) + C = -p + 2\mu v_y, \tag{9}$$
$$\rho x(-v_{yt} + v_y^2 - vv_{yy}) = D_x + E_y, \qquad \rho(v_t + vv_y) = E_x + C_y.$$

The obtained equalities should be supplemented by the relations

$$v_x = 0, \qquad p_x = 0, \qquad C_x = 0. \tag{10}$$

System (9), (10) is obviously overdetermined; therefore, the question of its compatibility arises. From the third equation in (9) and relation (10), it follows that $v_{yy}E = 0$. We consider both cases: $E = 0$ and $v_{yy} = 0$. In the first case, the second equation in (9) implies the equality $D_x = 0$, which prevents the extension of the set of exact solutions. In the second case, v is a linear function of y. For simplicity, we assume that this function is homogeneous in y:

$$v = -yk(t). \tag{11}$$

From (10) and the last two equations of system (9), it follows that the functions E and D are linear and quadratic functions of x, respectively. Next, the function E will be considered odd and the function D even in the variable x, so that

$$D = x^2 a(y,t) + d(y,t), \qquad E = xb(y,t), \qquad C = c(y,t), \qquad p = p(y,t). \tag{12}$$

Substitution of expressions (11) into Eq. (10) yields the equality

$$\tau(a_t - kya_y + 2ka) + a = 0, \qquad \tau(b_t - kyb_y + kb) + b = 0; \tag{13}$$

$$\rho(\dot{k} + k^2) = 2a + b_y; \tag{14}$$

$$\rho y(-\dot{k} + k^2) = b + c_y; \tag{15}$$

$$\tau(c_t - kyc_y) + c = -p - 2\mu k, \qquad \tau(d_t - kyd_y) = -p + 2\mu k. \tag{16}$$

Equations (13) and (14) form a closed subsystem for the functions $a(y,t)$, $b(y,t)$, and $k(t)$. If the solution of this subsystem is known, the function $c(y,t)$ is found by quadrature from Eq. (15), after which the function $p(y,t)$ is easily obtained from the first equation in (16). It remains to find the function $d(y,t)$ by solving the linear transport equation — the second equation in (16).

Let us consider system (13), (14), which contains three equations for three required functions and is formally closed. In fact, this system is overdetermined since the function k does not depend on y. Nevertheless, system (13), (14) is compatible. We first note that this system admits solutions in which the function a is even in the variable y and the function b is odd. A simple analysis shows that all such solutions are described by the formulas

$$b = \rho q y \exp{(-t/\tau)}, \qquad a = a(t), \tag{17}$$

where $q = \text{const}$; the functions $a(t)$ and $k(t)$ satisfy the system

$$\tau(\dot{a} + 2ka) + a = 0, \qquad \rho(\dot{k} + k^2) = 2a + \rho q \exp{(-t/\tau)}. \tag{18}$$

The extension of the family of exact solutions of system (1)–(3) obtained from its partially invariant solution (4), (5) can be treated as a new effect in the group analysis of differential equations. Another method of this extension is proposed in [10]. The group nature of solution (5), (8), (11), (12) of systems (1)–(3) is not clear. One might expect that it is a differential-invariant solution of the indicated system. It should be noted that the theory of differential-invariant solutions has only recently begun to be developed (see [11] and the references therein).

To system (18) we add the initial conditions

$$a(0) = a_0, \qquad k(0) = k_0. \tag{19}$$

The Cauchy problem (18), (19) requires a more detailed investigation. In the present paper, particular cases of this problem are considered. The first case corresponds to the value $q = 0$. In this case, system (18) is transformed to a dynamic system in a plane. Given the solution of the Cauchy problem (19) for the above-mentioned system

$$\tau(\dot{a} + 2ka) + a = 0, \qquad \rho(\dot{k} + k^2) = 2a, \tag{20}$$

it is possible to describe the motion of a viscoelastic strip with free boundaries in the x direction under the action of tension or compression forces distributed under a quadratic law. The functions $s(t)$ and $c(y,t)$ are defined by the equalities

$$s = s_0 \exp\Big(-\int\limits_0^t k(z)\,dz\Big), \qquad c = \frac{\rho}{2}\,(\dot{k} - k^2)(s^2 - y^2), \tag{21}$$

where s_0 is a positive constant. Then, Eq. (15) with $b = 0$ will be satisfied. It should be noted that the formulas specifying the velocity field and the stress distribution $P_{yy} = c(y,t)$ in the constructed solution do not include the viscosity μ although the components P_{xx} and p of the solution of system (1)–(3) depend on viscosity.

Let us define the flow region $S_T = \{x, y, t\ x \in \mathbb{R},\ |y| < s(t),\ 0 < t < T\}$. By virtue of equality (21) and $b = 0$, on the boundaries of the strip $y = \pm s(t)$, the conditions of no stress $P_{xy} = P_{yy} = 0$ are satisfied. From (11) and (21), it also follows that, on these lines, the kinematic condition $\dot{s} = v(s, t)$ is satisfied. This implies that the lines $y = \pm s(t)$ can be taken as free boundaries. From (11), (5), (17), and (21), it follows that the function v is an odd function y, and the functions u, a, and c are even functions. This guarantees the evenness of the function p determined from the first equation in (16). The evenness of the function d determined by solving the second equation in (16) is provided if its initial value $d(y, 0)$ possesses the evenness property. From the aforesaid, the solution in question can be treated as the motion of a viscoelastic medium in a strip with the symmetry line $y = 0$ and the free boundaries $y = \pm s(t)$. In this case, unlike in the example considered in Sec. 2, the source of motion is not only the initial velocity field but also the stress distributed along the x axis at the initial time.

Let us return to the Cauchy problem (19), (20). Depending on the values of a_0 and k_0, the solution of this problem is determined on the semi-axis $t > 0$ [in this case, $k = O(t^{-1})$ as $t \to \infty$ and the function a decreases exponentially] or collapses at the time $t = t^*$ (the nature of the collapse is described in Sec. 2). System (20) also has a steady-state solution $k = -1/(2\tau)$, $a = \rho/(8\tau^2)$, which corresponds to the following solution of the free-boundary problem for system (1)–(3):

$$\begin{gathered} u = -\frac{x}{2\tau}, \qquad v = \frac{y}{2\tau}, \qquad P_{xx} = \frac{\rho x^2}{8\tau^2} + d(y, t), \\ P_{xy} = 0, \qquad P_{yy} = \frac{\rho(y^2 - s^2)}{8\tau^2}, \qquad s = s_0 \exp\Big(\frac{t}{2\tau}\Big). \end{gathered} \tag{22}$$

Here the function d is determined by solving the linear equation of translation (16). This solution of the strip deformation problem is specific to an incompressible viscoelastic Maxwell medium and has no analog in the dynamics of a viscous incompressible fluid.

Let us consider another particular case of problem (18), (19): $a_0 = 0$. In this case, $a(t) \equiv 0$ and the problem reduces to the following:

$$t > 0\colon \quad \dot{k} + k^2 = q \exp(-t/\tau); \qquad t = 0\colon \quad k = k_0. \tag{23}$$

The type of solution of problem (23) depends on the sign of the constant q. If $q > 0$, then

$$k = q^{1/2} \exp\Big(-\frac{t}{2\tau}\Big) \frac{[q^{1/2} K_0'(z_0) - k_0 K_0(z_0)] I_0'(z) + [k_0 I_0(z_0) - q^{1/2} I_0'(z_0)] K_0'(z)}{[q^{1/2} K_0'(z_0) - k_0 K_0(z_0)] I_0(z) + [k_0 I_0(z_0) - q^{1/2} I_0'(z_0)] K_0(z)}. \tag{24}$$

Here $z(t) = 2\tau q^{1/2} \exp(-t/2\tau)$; $z_0 = z(0)$; $I_0(z)$ and $K_0(z)$ are Bessel functions of an imaginary argument of the first and second kind; prime denotes the derivative of these functions with respect to the argument. In the case $q < 0$, the solution of problem (23) has the form

$$k = |q|^{1/2} \exp\Big(-\frac{t}{2\tau}\Big) \frac{[|q|^{1/2} Y_0'(z_0) - k_0 Y_0(z_0)] J_0'(z) + [k_0 J_0(z_0) - |q|^{1/2} J_0'(z_0)] Y_0'(z)}{[|q|^{1/2} Y_0'(z_0) - k_0 Y_0(z_0)] J_0(z) + [k_0 J_0(z_0) - |q|^{1/2} J_0'(z_0)] Y_0(z)}, \tag{25}$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions of the first and second kind. As in the solution of the Cauchy problem (19), (20), two cases are possible: 1) the solution of problem (25) is extendable to the entire semi-axis $t > 0$; 2) a value $t^* > 0$ exists such that $|k| \to \infty$ as $t \to t^*$. In the first case, $k = O(t^{-1})$ as $t \to \infty$. This result follows from representation (24) and the asymptotic functions $I_0(z)$, $K_0(z)$ as $z \to 0$ (which corresponds to $t \to \infty$). The formal cause of the breakdown of the solution for $k_0 < 0$ is the presence of zeroes in the denominator in formula (25) due to a change in the sign of the functions $J_0(z)$ and $Y_0(z)$.

The solution of the Cauchy problem (23) there corresponds the following solution of the free-boundary problem for the initial system (1)–(3): the flow region is $S_T = \{x, y, t\colon x \in \mathbb{R},\ |y| < s(t),\ 0 < t < T\}$, the function $s(t)$ is given by the relation

$$s = s_0 \exp\Big(-\int_0^t k(z)\, dz\Big),$$

and the velocity and stress fields are given by the formulas

$$
\begin{gathered}
u = xk(t), \qquad v = -yk(t), \qquad P_{xx} = d(y,t), \qquad P_{xy} = \rho q x y \exp(-t/\tau), \\
P_{yy} = (\rho/2)[\dot{k} - k^2 - q \exp(-t/\tau)](s^2 - y^2).
\end{gathered}
\tag{26}
$$

Here the function $k(t)$ is defined by one of relations (24) and (25), and the function $d(y,t)$ is found by solving the second equation in (16) subject to the initial condition $d(y,0) = d_0(y)$. Here $d_0(y)$ is a function of the class $C^1[-s_0, s_0]$ that satisfies the evenness condition $[d_0'(0) = 0]$ and is generally arbitrary.

Solution (26) satisfies the kinematic condition and the condition of no normal stress on the boundary of the flow region S_T, but the shear stress on the lines $y = \pm s(t)$ does not vanish. This allows the following treatment of the solution obtained. At the initial time, the velocity field has the form $u = k_0 x$, $v = -k_0 y$ (in particular, the medium can be at rest: $k_0 = 0$) and the shear stresses dependent linearly on x is distributed on the strip boundaries $y = \pm s_0$. During motion this stress relaxes exponentially, and the motion enters an inertial regime similar to that considered in Sec. 2.

Conclusions. At first glance, solution (22) of the problem of deformation of a viscoelastic strip is paradoxical: the width of the strip increases with time despite the action of the tension forces. These forces, however, are overcome by the forces of inertia acting in the opposite direction according to the initial velocity distribution, and this eliminates the contradiction.

Physical implementation of solution (7) of the strip deformation problem and its analogs is complicated for two reasons. Truncation of the strip leads to the formulation of boundary conditions at its ends that are difficult to produce experimentally. The presence of even low gravity deforms the straight boundaries of a strip of finite length irrespective of how it is fixed in space. From the point of view of implementation, the solution of the problem of spread of a viscoelastic strip with a straight free boundary along a flat wall is more suitable. The mathematical formulation of this problem differs from that considered above in that the symmetry conditions on the lines $y = 0$ are replaced by the slip conditions $u = v = 0$ for $y = 0$ and $0 < t < T$. For the limiting case of a viscous incompressible fluid, this problem was studied in [8].

In conclusion, we discuss the question of the position of the incompressible medium model in viscoelastic continuum theory. From a thermodynamic point of view, an incompressible medium is defective (which, nevertheless, does not prevent the use of the equations of a viscous incompressible fluid to solve various applied problems). However, there are a large number of materials which have both viscous and elastic properties with different relaxation times of bulk and shear elastic stresses (see, for example, [12]). As regards water, different papers give different bulk-stress relaxation time τ' under pulse loading. In this case, it is more appropriate to speak of the range of relaxation times 10^{-8} sec $< \tau' < 10^{-6}$ sec. Meanwhile, the shear-stress relaxation time τ in water is 29.33 min, according to [3]. This implies that in the description of the evolution of such a medium at larger times from the specified initial state, it is necessary to study its behavior in the incompressible limit, which is an additional motivation to investigate the incompressible viscoelastic medium model.

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